THE CREPANT RESOLUTION CONJECTURE FOR TYPE A SURFACE SINGULARITIES

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Abstract. Let $\mathcal{X}$ be an orbifold with crepant resolution Y. The Crepant Resolution Conjectures of Ruan and Bryan–Graber assert, roughly speaking, that the quantum cohomology of $\mathcal{X}$ becomes isomorphic to the quantum cohomology of Y after analytic continuation in certain parameters followed by the specialization of some of these parameters to roots of unity. We prove these conjectures in the case where $\mathcal{X}$ is a surface singularity of type A. The key ingredient is mirror symmetry for toric orbifolds.

This Preprint is Obsolete

Please note that this preprint has been superseded by `arXiv:math/0702234v3` [17]. The material here, with various typos corrected, appears as Appendix A there.

1. Introduction

The small quantum cohomology of an orbifold $\mathcal{X}$ is a family of algebra structures on the Chen–Ruan orbifold cohomology $H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbb{C})$. This family depends on so-called *quantum parameters*, and encodes certain genus-zero Gromov–Witten invariants of $\mathcal{X}$. A long-standing conjecture of Ruan states that if $\mathcal{X}$ is an orbifold with coarse moduli space X and $Y \to X$ is a crepant resolution then the small quantum cohomology of Y becomes isomorphic to the small quantum cohomology of $\mathcal{X}$ after analytic continuation in the quantum parameters followed by specialization of some of the parameters to roots of unity. A refinement of this conjecture, proposed recently by Bryan and Graber [8], suggests that if $\mathcal{X}$ satisfies a Hard Lefschetz condition on orbifold cohomology then the Frobenius manifold structures defined by the quantum cohomology of $\mathcal{X}$ and of Y coincide after analytic continuation and specialization of parameters (see also [16] for a Hard Lefschetz condition). This is a stronger assertion: that the *big* quantum cohomology of Y coincides with that of $\mathcal{X}$ after analytic continuation plus specialization, via a linear isomorphism which preserves the (orbifold) Poincaré pairing. In this note we prove these conjectures in the case where $\mathcal{X}$ is the A_{n-1} surface singularity $\left[\mathbb{C}^2/\mu_n\right]$ and Y is its crepant resolution. In fact we prove a more precise statement, Theorem 1 below, which also identifies an isomorphism and the roots of unity to which the quantum parameters of Y are specialized. We learned this statement from Jim Bryan [6; 8, Conjecture 3.1] and Fabio Perroni [29, Conjecture 1.9; 30].

Our proof of Theorem 1 is based on mirror symmetry for toric orbifolds. By mirror symmetry we mean the fact, first observed by Candelas *et al.* [11], that one can compute virtual numbers of rational curves in a manifold or orbifold $\mathcal{X}$ — *i.e.* certain Gromov–Witten invariants of $\mathcal{X}$ — by solving Picard–Fuchs equations. Following Givental, we will formulate this precisely as a relationship between a

cohomology-valued generating function for genus-zero Gromov–Witten invariants, called the *J-function of $\mathcal{X}$*, and a cohomology-valued solution to the Picard–Fuchs equations called the *I-function of $\mathcal{X}$*. This relationship is Proposition 2 in Section 4. After describing the toric structures of $\mathcal{X}$ and Y in Section 2 and fixing notation for cohomology and quantum cohomology in Section 3, we explain in Section 4 how to extract the quantum products for $\mathcal{X}$ and Y from the Picard–Fuchs equations. Once we understand this, Theorem 1 follows easily: the proof is at the end of Section 4.

A number of cases of Theorem 1 were already known. Ruan's Crepant Resolution Conjecture was established for surface singularities of type A_1 and A_2 by Perroni [29]. Theorem 1 was proved in the A_1 case by Bryan-Graber [8], in the A_2 case by Bryan–Graber–Pandharipande [9], and in the A_3 case by Bryan–Jiang [10]. Davesh Maulik has computed the genus-zero Gromov–Witten potential of the type A surface singularity $\mathcal{X} = \left[\mathbb{C}^2/\mu_n\right]$ for all n (as well as certain higher-genus Gromov–Witten invariants of $\mathcal{X}$) and the reduced genus-zero Gromov–Witten potential of the crepant resolution Y [27]; Theorem 1 should follow from this. The quantum cohomology of the crepant resolutions of type ADE surface singularities has been computed by Bryan–Gholampour [7]. Skarke [31] and Hosono [23] have studied the A_n case from a point of view very similar to ours, as part of their investigations of homological mirror symmetry.

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2. $\mathcal{X}$ and Y as Toric Orbifolds

$\mathcal{X}$ is the toric orbifold corresponding to the fan[1] in Figure 1(a) and Y is the toric manifold corresponding to the fan in Figure 1(b). Background material on toric manifolds and orbifolds can be found in [2, Chapter VII].

There is an exact sequence

$$0 \longrightarrow \mathbb{Z}^{n-1} \xrightarrow{M^{\mathrm{T}}} \mathbb{Z}^{n+1} \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n \end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0,$$

and hence we can represent the Gale dual of the right-hand map by

$$\mathbb{Z}^{n+1} \xrightarrow{M} \mathbb{Z}^{n-1},$$

[1] $\mathcal{X}$ is also the toric Deligne–Mumford stack [4] corresponding to the stacky fan in Figure 1(a).

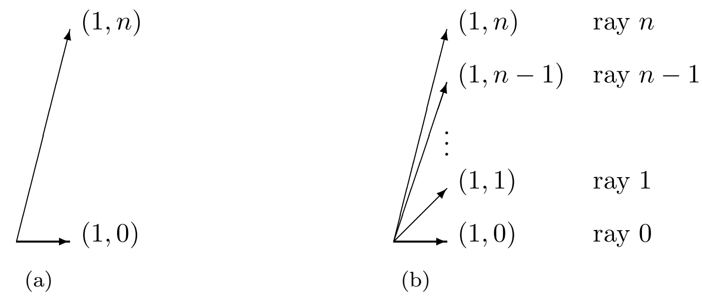

FIGURE 1. (a) The fan for $\mathcal{X}$. (b) The fan for Y.

where

$$M = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

Certain faces of the positive orthant $(\mathbb{R}_{\geq 0})^{n+1} \subset \mathbb{R}^{n+1}$ project via M to codimension-1 subsets of $\mathbb{R}^{n-1}$. The image of the positive orthant is divided by these subsets into chambers, which are the maximal cones of a fan in $\mathbb{R}^{n-1}$ called the *secondary fan* of Y. Chambers in the secondary fan correspond to toric partial resolutions of $\mathcal{X}$. A chamber K corresponds to a fan Σ with rays some subset of the rays of the fan for Y, as follows. Number the rays of the fan for Y as shown in Figure 1(b). For a subset $\sigma \subset \{0, 1, \ldots, n\}$, let us write $\bar{\sigma}$ for the complement $\{0, 1, \ldots, n\} \setminus \sigma$, $\mathbb{R}^\sigma$ for the corresponding co-ordinate subspace of $\mathbb{R}^{n+1}$, and say that σ covers K iff $K \subset M(\mathbb{R}^\sigma)$. The fan Σ corresponding to the chamber K is defined by

$$\sigma \in \Sigma \iff \bar{\sigma} \text{ covers } K;$$

the chamber K corresponding to the fan Σ is

$$\bigcap_{\sigma \in \Sigma} M\left(\mathbb{R}^{\bar{\sigma}}\right).$$

We will concentrate on two chambers: $K_{\mathcal{X}}$, with rays given by the middle $n-1$ columns of M, and K_Y with rays given by the standard basis vectors for $\mathbb{R}^{n-1}$. $K_{\mathcal{X}}$ corresponds to the toric orbifold $\mathcal{X}$ and K_Y corresponds to the toric manifold Y.

Let $\mathcal{M}_{\mathrm{sec}}$ be the toric orbifold corresponding to the secondary fan of Y. As $K_{\mathcal{X}}$ and K_Y are simplicial, they give co-ordinate patches on $\mathcal{M}_{\mathrm{sec}}$: the co-ordinates $x_1, \ldots, x_{n-1}$ from $K_{\mathcal{X}}$ and $y_1, \ldots, y_{n-1}$ from K_Y are related by

$$y_i = \begin{cases} x_1^{-2} x_2 & i = 1 \\ x_{i-1} x_i^{-2} x_{i+1} & 1 < i < n-1 \\ x_{n-2} x_{n-1}^{-2} & i = n-1. \end{cases} \tag{1a}$$

More precisely, $x_1, \ldots, x_{n-1}$ are multi-valued and the co-ordinate patch $\mathcal{M}_{\mathrm{sec}}(K_{\mathcal{X}})$ corresponding to the cone $K_{\mathcal{X}}$ is given by the uniformizing system:

$$\mathcal{M}_{\mathrm{sec}}(K_{\mathcal{X}}) \cong \mathbb{C}^{n-1}/\mu_n, \quad (x_1, x_2, \ldots, x_{n-1}) \sim (c x_1, c^2 x_2, \ldots, c^{n-1} x_{n-1}) \text{ for } c \in \mu_n.$$

The *B-model moduli space* $\mathcal{M}_B$ is the open subset $\mathbb{C} \times \mathcal{M}_{\text{sec}}(K_{\mathcal{X}})$ of $\mathbb{C} \times \mathcal{M}_{\text{sec}}$. Denote by x_0 or y_0 the co-ordinate on the first factor $\mathbb{C}$ of $\mathbb{C} \times \mathcal{M}_{\text{sec}}$, so that

$$x_0 = y_0. \tag{1b}$$

We will refer to the point $(x_0, x_1, \dots, x_{n-1}) = (0, 0, \dots, 0)$ as the *large-radius limit point for* $\mathcal{X}$ and the point $(y_0, y_1, \dots, y_{n-1}) = (0, 0, \dots, 0)$ as the *large-radius limit point for* Y. The co-ordinates x_i and y_j are related to each other by (1), so that $y_0, y_1, \dots, y_{n-1}$ are co-ordinates on the patch $\mathbb{C} \times (\mathbb{C}^\times)^{n-1} \subset \mathbb{C} \times \mathcal{M}_{\text{sec}}(K_{\mathcal{X}}) = \mathcal{M}_B$ where each of $x_1, x_2, \dots, x_{n-1}$ is non-zero.

Remark. In what follows the first factor of $\mathcal{M}_B$, which has co-ordinates x_0 or y_0, will play a rather different role than the second factor. The first factor will correspond under mirror symmetry to $H^0_{\text{orb}}(\mathcal{X}) \subset H^\bullet_{\text{orb}}(\mathcal{X})$ or $H^0(Y) \subset H^\bullet(Y)$, and the second factor will correspond to $H^2_{\text{orb}}(\mathcal{X}) \subset H^\bullet_{\text{orb}}(\mathcal{X})$ or $H^2(Y) \subset H^\bullet(Y)$.

Remark. It would be more honest to define the B-model moduli space as the product of $\mathbb{C}$ with the open subset of $\mathcal{M}_{\text{sec}}$ on which the GKZ system associated to Y is non-singular. This set is slightly smaller than $\mathcal{M}_B$, as it does not contain the discriminant locus of $W_{\mathcal{X}}$ or W_Y which appears below (in the proof of Proposition 4).

The presentations of $\mathcal{X}$ as a toric orbifold and Y as a toric variety allow us to write $\mathcal{X}$ and Y as quotients of open sets $\mathcal{U}_{\mathcal{X}}, \mathcal{U}_Y \subset \mathbb{C}^{n+1}$ by $(\mathbb{C}^\times)^{n-1}$. The action of $T = (\mathbb{C}^\times)^2$ on $\mathbb{C}^{n+1}$ given by

$$(a_0, a_1, \dots, a_n) \xmapsto{(s,t)} (sa_0, a_1, a_2, \dots, a_{n-1}, ta_n) \tag{2}$$

descends to give T-actions on $\mathcal{X}$, X, and Y, and the crepant resolution $Y \to X$ is T-equivariant. The T-fixed locus on Y is the exceptional divisor. The T-action on $\mathcal{X} = [\mathbb{C}^2/\mu_n]$ coincides with that induced by the standard action of T on $\mathbb{C}^2$, so the T-fixed locus on $\mathcal{X}$ is the $B\mu_n$ at the origin. We write $H^\bullet_T(\{\text{pt}\}) = \mathbb{C}[\lambda_1, \lambda_2]$ where λ_i is Poincaré-dual to a hyperplane in the ith factor of $(\mathbf{CP}^\infty)^2 \simeq BT$.

3. Cohomology and Quantum Cohomology

We will assume familiarity with quantum orbifold cohomology, referring the reader to [18, Section 2] for a brief overview and the original sources [1, 13] for a detailed exposition. We will assume also familiarity with the work of Bryan–Graber [8], and in particular with their enhanced notion of the degree of a stable map to an orbifold ("degree in the twisted sectors"). Our notation will be compatible with that in [8].

The T-equivariant orbifold cohomology $H^\bullet_{T,\text{orb}}(\mathcal{X}; \mathbb{C})$ is the T-equivariant cohomology of the inertia stack $\mathcal{IX}$. $\mathcal{IX}$ has components $\mathcal{X}_0$, $\mathcal{X}_1$, …,$\mathcal{X}_{n-1}$, where

$$\mathcal{X}_k = \left[\left(\mathbb{C}^2\right)^g / \mu_n \right] \qquad \text{with } g = \exp\left(2k\pi\sqrt{-1}/n\right) \in \mu_n.$$

We have

$$\begin{aligned} &\mathcal{X}_k = \left[\mathbb{C}^2/\mu_n\right] && \text{age} = 0 && \text{if } k = 0, \\ &\mathcal{X}_k = B\mu_n && \text{age} = 1 && \text{otherwise.} \end{aligned}$$

Let δ_i be the fundamental class of $\mathcal{X}_i$, $0 \le i < n$; this gives a $\mathbb{C}[\lambda_1, \lambda_2]$-basis for $H^\bullet_{T,\text{orb}}(\mathcal{X}; \mathbb{C})$. The canonical involution I on $\mathcal{IX}$ fixes $\mathcal{X}_0$ and exchanges $\mathcal{X}_i$ with

$\mathcal{X}_{n-i}$, $1 \leq i < n$. As I is age-preserving, $H^\bullet_{\mathrm{orb}}(\mathcal{X};\mathbb{C})$ satisfies Hard Lefschetz [8, Definition 1.1; 20].

The cone K_Y is the Kähler cone for Y and its rays determine a basis $\gamma_1, \ldots, \gamma_{n-1}$ for $H^2(Y;\mathbb{Z})$. The dual basis $\beta_1, \ldots, \beta_{n-1}$ for $H_2(Y;\mathbb{Z})$ is *positive* in the sense of [8, Section 1.2]. If we define $\gamma_0 = 1$ and choose lifts of $\gamma_1, \ldots, \gamma_{n-1}$ to T-equivariant cohomology then γ_i, $0 \leq i < n$, is an $\mathbb{C}[\lambda_1, \lambda_2]$-basis for $H^\bullet_T(Y;\mathbb{C})$. We choose a standard equivariant lift of each $\gamma \in H^2(Y;\mathbb{Z})$ in the following way. There is a unique representation ρ_γ of $(\mathbb{C}^\times)^{n-1}$ such that γ is the first Chern class of the line bundle

$$L_\gamma := \mathcal{U}_Y \times_{\rho_\gamma} \mathbb{C} \longrightarrow \mathcal{U}_Y/(\mathbb{C}^\times)^{n-1} = Y.$$

This line bundle L_γ admits a T-action such that T acts on $\mathcal{U}_Y$ via (2) and acts trivially on the $\mathbb{C}$ factor, and the lift $\gamma \in H^2_T(Y;\mathbb{Z})$ is the T-equivariant first Chern class of L_γ. The columns of M, together with the action (2), define elements $\omega_j \in H^2_T(Y;\mathbb{C})$, $0 \leq j \leq n$, where

$$\omega_j = \begin{cases} \lambda_1 + \gamma_1 & j = 0 \\ -2\gamma_1 + \gamma_2 & j = 1 \\ \gamma_{j-1} - 2\gamma_j + \gamma_{j+1} & 1 < j < n-1 \\ \gamma_{n-2} - 2\gamma_{n-1} & j = n-1 \\ \lambda_2 + \gamma_{n-1} & j = n. \end{cases}$$

The class ω_i is the T-equivariant Poincaré dual of the toric divisor given in coordinates (2) by $a_i = 0$. We have

$$H^\bullet_T(Y;\mathbb{C}) = \mathbb{C}[\lambda_1, \lambda_2, \gamma_1, \ldots, \gamma_{n-1}] / \langle \omega_i \omega_j : i - j > 1 \rangle .$$

$\mathcal{X}$ and Y are non-compact but nonetheless, as discussed in [8], one can define (orbifold) Poincaré pairings on the localized T-equivariant (orbifold) cohomology groups

$$H(\mathcal{X}) := H^\bullet_{T,\mathrm{orb}}(\mathcal{X};\mathbb{C}) \otimes \mathbb{C}(\lambda_1, \lambda_2) \qquad \text{and} \qquad H(Y) := H^\bullet_T(Y;\mathbb{C}) \otimes \mathbb{C}(\lambda_1, \lambda_2)$$

using the Bott residue formula. These pairings take values in $\mathbb{C}(\lambda_1, \lambda_2)$, and are non-degenerate. Similarly, even though some moduli spaces of stable maps to $\mathcal{X}$ or Y are non-compact, the T-fixed loci on these moduli spaces are compact and so we can still define $\mathbb{C}(\lambda_1, \lambda_2)$-valued Gromov–Witten invariants of $\mathcal{X}$ and Y using the virtual localization formula of Graber–Pandharipande [22]. For $\alpha_1, \ldots, \alpha_n \in H(Y)$, $\beta \in H_2(Y;\mathbb{Z})$, and $i_1, \ldots, i_n \geq 0$, we set

$$\left\langle \alpha_1 \psi_1^{i_1}, \ldots, \alpha_n \psi_n^{i_n} \right\rangle_\beta^Y = \int_{\left[\overline{M}_{0,n}(Y,\beta)\right]^{\mathrm{vir}}} \prod_{j=1}^n \mathrm{ev}_j^\star \alpha_j \cdot \psi_j^{i_j}.$$

Here $\psi_1, \ldots, \psi_n$ are the universal cotangent line classes on the moduli space $\overline{M}_{0,n}(Y,\beta)$ of genus-zero n-pointed stable maps to Y of degree β; the integral is defined by localization to the T-fixed substack, as in [22, Section 4] or [15, Section 3.1]. Bryan and Graber define a set of *effective classes in the orbifold Neron–Severi group* of $\mathcal{X}$, and for each effective class $\widehat{\beta}$ describe an associated moduli space $\overline{M}_{0,n}(\mathcal{X},\widehat{\beta})$ of genus-zero n-pointed stable maps to $\mathcal{X}$. As the notation suggests, one can think of $\widehat{\beta}$ as recording some sort of degree of a stable map to $\mathcal{X}$: an effective class consists of a non-negative integer $\widehat{\beta}(i)$ for each inertia component $\mathcal{X}_i$, $1 \leq i < n$, and a

stable map in $\overline{M}_{0,n}(\mathcal{X},\widehat{\beta})$ carries $\widehat{\beta}(i)$ extra unordered marked points which are constrained to map to $\mathcal{X}_i$. We write

$$\big\langle \alpha_1\psi_1^{i_1},\ldots,\alpha_n\psi_n^{i_n}\big\rangle^{\mathcal{X}}_{\widehat{\beta}} = \int_{\left[\overline{M}_{0,n}(\mathcal{X},\widehat{\beta})\right]^{\mathrm{vir}}} \prod_{j=1}^{n} \mathrm{ev}_j^{\star}\alpha_j \cdot \psi_j^{i_j},$$

where $\alpha_1,\ldots,\alpha_n \in H(\mathcal{X})$; $i_1,\ldots,i_n \geq 0$; $\psi_1,\ldots,\psi_n$ are the universal cotangent line classes on $\overline{M}_{0,n}(\mathcal{X},\widehat{\beta})$; and the integral is once again defined by virtual localization. These integrals are in fact certain local Gromov–Witten invariants [14].

The genus-zero Gromov–Witten invariants defined here assemble to give associative quantum products on $H(\mathcal{X})$ and $H(Y)$. The small quantum product for $\mathcal{X}$ is the $\mathbb{C}(\lambda_1,\lambda_2)$-algebra defined by

$$\delta_i \star \delta_j = \sum_{k=0}^{n-1} \langle \delta_i,\delta_j,\delta_k\rangle^{\mathcal{X}}_{\widehat{0}}\, \delta^k. \tag{3}$$

Here $\widehat{0}$ is the orbifold Neron-Severi class with $\widehat{0}(i) = 0$, $1 \leq i < n$, and $\{\delta^i\}$ denotes the basis dual to $\{\delta_i\}$ under the orbifold Poincaré pairing. The small quantum product for $\mathcal{X}$ coincides with the Chen–Ruan or orbifold cup product [12]. The small quantum product for Y is the family of $\mathbb{C}(\lambda_1,\lambda_2)$-algebras, depending on parameters $q_1,\ldots,q_{n-1}$, defined by

$$\gamma_i \star \gamma_j = \sum_{\beta}\sum_{k=0}^{n-1} \langle \gamma_i,\gamma_j,\gamma_k\rangle^{Y}_{\beta}\, q_1^{d_1}\cdots q_{n-1}^{d_{n-1}}\gamma^k. \tag{4}$$

The sum here is over classes $\beta = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$ with each $d_i \geq 0$, and $\{\gamma^i\}$ denotes the basis dual to $\{\gamma_i\}$ under the Poincaré pairing. It follows from the discussion below that the right-hand side of (4) defines an analytic function of $q_1,\ldots,q_{n-1}$ in some neighbourhood of the origin. Ruan's conjecture asserts that the small quantum cohomology algebra $(H(Y),\star)$ becomes isomorphic to $(H(\mathcal{X}),\star)$ after analytic continuation in the q_i followed by setting the q_i equal to certain roots of unity.

Bryan and Graber's refinement of Ruan's conjecture involves big quantum cohomology. The big quantum cohomology of $\mathcal{X}$ is the family of $\mathbb{C}(\lambda_1,\lambda_2)$-algebras parametrized by $u \in H(\mathcal{X})$, $u = u_0\delta_0 + u_1\delta_1 + \cdots + u_{n-1}\delta_{n-1}$, defined by

$$\delta_i \underset{\mathrm{big}}{\star} \delta_j = \sum_{\widehat{\beta}}\sum_{k=0}^{n-1} \langle \delta_i,\delta_j,\delta_k\rangle^{\mathcal{X}}_{\widehat{\beta}}\, u_1^{\widehat{\beta}(1)}\cdots u_{n-1}^{\widehat{\beta}(n-1)}\delta^k. \tag{5}$$

The sum here is over effective classes $\widehat{\beta}$. The big quantum cohomology of Y is the family of $\mathbb{C}(\lambda_1,\lambda_2)$-algebras parametrized by $t \in H(Y)$, $t = t_0\gamma_0 + t_1\gamma_1 + \cdots + t_{n-1}\gamma_{n-1}$, defined by

$$\gamma_i \underset{\mathrm{big}}{\star} \gamma_j = \sum_{\beta}\sum_{k=0}^{n-1} \langle \gamma_i,\gamma_j,\gamma_k\rangle^{Y}_{\beta}\, e^{d_1t_1+\cdots+d_{n-1}t_{n-1}}\gamma^k. \tag{6}$$

The sum here is over classes $\beta = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$ with each $d_i \geq 0$. Together with the (orbifold) Poincaré pairings, the big quantum cohomology algebras define Frobenius manifolds[2] based on $H(\mathcal{X})$ and $H(Y)$. The Bryan–Graber version of the Crepant Resolution Conjecture asserts that these Frobenius manifolds coincide

[2]These Frobenius manifolds are defined over the field $\mathbb{C}(\lambda_1,\lambda_2)$.

after analytic continuation in the t_i and an appropriate change-of-variables. This is our main result.

Theorem 1. *The big quantum products* (5) *for* $\mathcal{X}$ *and* (6) *for* Y *coincide after analytic continuation in the* t_i*, the affine-linear change-of-variables*

$$t_i = \begin{cases} u_0, & i = 0 \\ -\frac{2\pi\sqrt{-1}}{n} + \sum_{j=1}^{n-1} L_{ij} u_j, & i > 0, \end{cases}$$

where

$$L_{ij} = \frac{\zeta^{2ij}\left(\zeta^{-j} - \zeta^{j}\right)}{n}, \qquad \zeta = \exp\left(\frac{\pi\sqrt{-1}}{n}\right),$$

and the linear isomorphism

$$\begin{aligned} L : H(\mathcal{X}) &\to H(Y) \\ \delta_0 &\mapsto \gamma_0, \\ \delta_j &\mapsto \sum_{i=1}^{n-1} L_{ij}\gamma_i, \qquad 1 \leq j < n. \end{aligned} \tag{7}$$

Furthermore, the isomorphism (7) *matches the Poincaré pairing on* $H(Y)$ *with the orbifold Poincaré pairing on* $H(\mathcal{X})$.

Theorem 1 establishes Conjecture 3.1 in [8] for the case of polyhedral and binary polyhedral groups of type A, and also Conjecture 1.9 in [29]. The path along which analytic continuation is taken is described after Proposition 4 below.

To pass from the big quantum cohomology algebras of $\mathcal{X}$ and Y to the small quantum cohomology algebras, set $u_i = 0$, $e^{t_i} = q_i$, $1 \leq i < n$.

Corollary. *The small quantum products* (3) *for* $\mathcal{X}$ *and* (4) *for* Y *coincide after analytic continuation in the* q_i*, the linear isomorphism* (7)*, and the specialization*

$$q_i = \exp\left(-\frac{2\pi\sqrt{-1}}{n}\right), \qquad 1 \leq i < n.$$

Remark. It would be more conventional to relate the big and small quantum cohomology algebras of Y by the change of variables $q_i = Q_i e^{t_i}$, $1 \leq i < n$, where Q_i is an element of a formal power series ring (or Novikov ring) introduced to ensure convergence of the product (see *e.g.* [19, Section 8.5.1]); the resulting product would then depend on two families of variables $t_0, \ldots, t_{n-1}$ and $Q_1, \ldots, Q_{n-1}$. We will not do this. Firstly this is because there are no convergence problems here — the right-hand side of (6) defines an analytic function of $t_0, \ldots, t_{n-1}$ on an appropriate domain — and secondly our choice makes clear how the specialization $q_i = c_i$ of quantum parameters to roots of unity arises: it just reflects the affine-linear identification of flat co-ordinates

$$t_i = \log c_i + \sum_{j=1}^{n-1} L_{ij} u_j, \qquad 1 \leq i < n.$$

4. Mirror Symmetry

As discussed in the Introduction, by mirror symmetry we mean the fact that one can compute certain genus-zero Gromov–Witten invariants of $\mathcal{X}$ and Y by solving Picard–Fuchs equations. In this Section we make this precise. We introduce two cohomology-valued generating functions for genus-zero Gromov–Witten invariants, called the *J-functions* of $\mathcal{X}$ and Y, and two cohomology-valued solutions to the Picard–Fuchs equations called the *I-functions* of $\mathcal{X}$ and Y. The relationship between the I-functions and the J-functions is given in Proposition 2 below. We then describe how to extract the quantum products (5) and (6) from the Picard–Fuchs equations, and finally explain how this implies Theorem 1.

4.1. **The I-Function and the J-Function.** The J-function $J_{\mathcal{X}}(u,z)$ of $\mathcal{X}$ is defined to be

$$e^{u_0/z}\left(z\delta_0+u_1\delta_1+\cdots+u_{n-1}\delta_{n-1}+\sum_{\widehat{\beta}}\sum_{k=0}^{n-1}\left\langle\frac{\delta_k}{z-\psi_1}\right\rangle_{\widehat{\beta}}^{\mathcal{X}}u_1^{\widehat{\beta}(1)}\cdots u_{n-1}^{\widehat{\beta}(n-1)}\delta^k\right).$$

The sum here is over effective classes $\widehat{\beta}$, and we expand $1/(z-\psi_1)$ as $\sum_m \psi_1^m/z^{m+1}$. $J_{\mathcal{X}}(u,z)$ is a function of $u\in H(\mathcal{X})$, $u=u_0\delta_0+\cdots+u_{n-1}\delta_{n-1}$, which takes values in $H(\mathcal{X})\otimes\mathbb{C}(\!(z^{-1})\!)$. It is defined and analytic in an open subset of $H(\mathcal{X})$ where $|u_1|,\ldots,|u_{n-1}|$ are sufficiently small; this follows from Proposition 2 below.

The J-function $J_Y(t,z)$ of Y is

$$e^{t_0/z}e^{(t_1\gamma_1+\cdots+t_{n-1}\gamma_{n-1})/z}\left(z\gamma_0+\sum_{\beta}\sum_{k=0}^{n-1}\left\langle\frac{\gamma_k}{z-\psi_1}\right\rangle_{\beta}^{Y}e^{d_1t_1+\cdots+d_{n-1}t_{n-1}}\gamma^k\right),$$

where the sum is over $\beta=d_1\beta_1+\cdots+d_{n-1}\beta_{n-1}$ with each $d_i\geq 0$. $J_Y(t,z)$ is a function of $t\in H(Y)$, $t=t_0\gamma_0+\cdots+t_{n-1}\gamma_{n-1}$, which takes values in $H(Y)\otimes\mathbb{C}(\!(z^{-1})\!)$. It is defined and analytic in an open subset of $H(Y)$ where $\Re(t_i)\ll 0$, $1\leq i<n$; this again follows from Proposition 2.

Given a class $\widehat{\beta}$ in the orbifold Neron–Severi group of $\mathcal{X}$, or in other words given a sequence $\widehat{\beta}(1),\ldots,\widehat{\beta}(n-1)$ of integers, it will be convenient to set

$$\widehat{\beta}(0)=-\frac{1}{n}\sum_{k=1}^{n-1}(n-k)\widehat{\beta}(k),\qquad \widehat{\beta}(n)=-\frac{1}{n}\sum_{k=1}^{n-1}k\widehat{\beta}(k),\qquad i(\widehat{\beta})=n\left\langle-\widehat{\beta}(n)\right\rangle,$$

where $\langle r\rangle$ denotes the fractional part of a rational number r. The I-function $I_{\mathcal{X}}(x,z)$ of $\mathcal{X}$ is defined to be

$$ze^{x_0/z}\sum_{\widehat{\beta}}\frac{1}{z^{\widehat{\beta}(1)+\cdots+\widehat{\beta}(n-1)}}\prod_{\substack{r:\widehat{\beta}(0)<r\leq 0\\ \langle r\rangle=\langle\widehat{\beta}(0)\rangle}}(\lambda_1+rz)\prod_{\substack{s:\widehat{\beta}(n)<s\leq 0\\ \langle s\rangle=\langle\widehat{\beta}(n)\rangle}}(\lambda_2+sz)\,\frac{x_1^{\widehat{\beta}(1)}\cdots x_{n-1}^{\widehat{\beta}(n-1)}}{\widehat{\beta}(1)!\cdots\widehat{\beta}(n-1)!}\,\delta_{i(\widehat{\beta})};$$

the sum here is over effective classes $\widehat{\beta}$. This is a function of $x=(x_0,\ldots,x_{n-1})\in\mathcal{M}_B$, $z\in\mathbb{C}^\times$, and $\lambda_1,\lambda_2\in\mathbb{C}$ which takes values in $H^\bullet_{T,\mathrm{orb}}(\mathcal{X};\mathbb{C})$. Each component of $I_{\mathcal{X}}(x,z)$ with respect to the basis $\{\delta_i\}$ is an analytic function of $(x,z,\lambda_1,\lambda_2)$ defined in a domain where $|x_1|,\ldots,|x_n|$ are sufficiently small and $x_0,z,\lambda_1,\lambda_2$ are arbitrary. By taking a Laurent expansion at $z=\infty$ we can regard $I_{\mathcal{X}}(x,z)$ as an analytic function of (x,λ_1,λ_2) which takes values in $H(\mathcal{X})\otimes\mathbb{C}(\!(z^{-1})\!)$. $I_{\mathcal{X}}(x,z)$

satisfies a system of Picard–Fuchs equations, as follows. Define differential operators $\sqsupset_i = z x_i \frac{\partial}{\partial x_i}$, $1 \leq i < n$, and

$$\sqsupset_0 = \lambda_1 - \frac{1}{n}\sum_{k=1}^{n-1}(n-k) z x_k \frac{\partial}{\partial x_k}, \qquad \sqsupset_n = \lambda_2 - \frac{1}{n}\sum_{k=1}^{n-1} k z x_k \frac{\partial}{\partial x_k}.$$

Then

$$\left(\prod_{j:\widehat{\beta}(j)>0}\ \prod_{m=0}^{\widehat{\beta}(j)-1}(\sqsupset_j - mz)\right) I_{\mathcal{X}}(x,z) = \tag{8a}$$
$$x_1^{\widehat{\beta}(1)} \cdots x_{n-1}^{\widehat{\beta}(n-1)} \left(\prod_{j:\widehat{\beta}(j)<0}\ \prod_{m=0}^{-\widehat{\beta}(j)-1}(\sqsupset_j - mz)\right) I_{\mathcal{X}}(x,z).$$

for each orbifold Neron–Severi class $\widehat{\beta}$ such that $i(\widehat{\beta}) = 0$, and

$$z\frac{\partial}{\partial x_0} I_{\mathcal{X}}(x,z) = I_{\mathcal{X}}(x,z). \tag{8b}$$

The I-function of Y is

$$I_Y(y,z) = z\, e^{y_0/z} y_1^{\gamma_1/z}\cdots y_{n-1}^{\gamma_{n-1}/z}\sum_{\beta}\prod_{j=0}^{n}\frac{\prod_{m\leq 0}(\omega_j+mz)}{\prod_{m\leq D_j(\beta)}(\omega_j+mz)} y_1^{d_1}\cdots y_{n-1}^{d_{n-1}},$$

where $y_i^{\gamma_i/z} = \exp(\gamma_i \log y_i/z)$, the sum is over $\beta = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$ with each $d_i \geq 0$, and

$$D_j(\beta) = \begin{cases} d_1 & j=0 \\ -2d_1 + d_2 & j=1 \\ d_{j-1} - 2d_j + d_{j+1} & 1<j<n-1 \\ d_{n-2} - 2d_{n-1} & j=n-1 \\ d_{n-1} & j=n. \end{cases}$$

$I_Y(y,z)$ is a multi-valued function of $y = (y_0,\ldots,y_{n-1}) \in \mathcal{M}_B$, $z \in \mathbb{C}^\times$, and $\lambda_1, \lambda_2 \in \mathbb{C}$ which takes values in $H_T^\bullet(Y;\mathbb{C})$. Each component of $I_Y(y,z)$ with respect to the basis $\{\gamma_i\}$ is a multi-valued analytic function of $(y,z,\lambda_1,\lambda_2)$ defined in a domain where $|y_1|,\ldots,|y_{n-1}|$ are sufficiently small, $|z| > \max(|\lambda_1|,|\lambda_2|)$, and y_0 is arbitrary. By taking a Laurent expansion at $z=\infty$ we can regard $I_Y(y,z)$ as a multi-valued analytic function of (y,λ_1,λ_2) which takes values in $H(Y)\otimes\mathbb{C}(\!(z^{-1})\!)$. It also satisfies a system of Picard–Fuchs equations. Define differential operators

$$\daleth_j = \begin{cases} \lambda_1 + z y_1 \frac{\partial}{\partial y_1} & j=0 \\ -2zy_1\frac{\partial}{\partial y_1} + zy_2\frac{\partial}{\partial y_2} & j=1 \\ zy_{j-1}\frac{\partial}{\partial y_{j-1}} - 2zy_j\frac{\partial}{\partial y_j} + zy_{j+1}\frac{\partial}{\partial y_{j+1}} & 1<j<n-1 \\ zy_{n-2}\frac{\partial}{\partial y_{n-2}} - 2zy_{n-1}\frac{\partial}{\partial y_{n-1}} & j=n-1 \\ \lambda_2 + zy_{n-1}\frac{\partial}{\partial y_{n-1}} & j=n. \end{cases}$$

Then

$$\left(\prod_{j : D_j(\beta) > 0} \prod_{m=0}^{D_j(\beta)-1} (\daleth_j - mz) \right) I_Y(y,z) = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}} \left(\prod_{j : D_j(\beta) < 0} \prod_{m=0}^{-D_j(\beta)-1} (\daleth_j - mz) \right) I_Y(y,z) \tag{9a}$$

for every $\beta = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$, and

$$z \frac{\partial}{\partial y_0} I_Y(y,z) = I_Y(y,z). \tag{9b}$$

The Picard–Fuchs systems (8) for $\mathcal{X}$ and (9) for Y coincide under the co-ordinate change (1). Thus there is a global system of Picard–Fuchs equations — a $\mathcal{D}$-module over all of $\mathcal{M}_B$ — which gives (8) near the large-radius limit point for $\mathcal{X}$ and (9) near the large-radius limit point for Y. This global nature of the Picard–Fuchs system will play a key role in what follows.

By mirror symmetry, we mean the following.

Proposition 2.

(1) *$I_{\mathcal{X}}(x,z)$ and $J_{\mathcal{X}}(u,z)$ coincide after a change of variables expressing u in terms of x.*
(2) *$I_Y(y,z)$ and $J_Y(t,z)$ coincide after a change of variables expressing t in terms of y.*

Proof. Part (1) is equation 23 in [17]. To see this, set t^i there to x_i, $0 \le i < n$; k_i there to $\widehat{\beta}(i)$, $1 \le i < n$; τ^i there to u_i, $0 \le i < n$; λ_1 there to λ_2 here and *vice versa*. Then $I^{\mathrm{tw}}(t,z)$ there coincides with $I_{\mathcal{X}}(x,z)$ here and $J^{\mathrm{tw}}(\tau,z)$ there coincides with $J_{\mathcal{X}}(u,z)$ here.

The argument that proves Theorem 0.2 in [21] also proves part (2) here. Theorem 0.2 as stated only applies to compact semi-positive toric manifolds, but the proof applies essentially without change to the non-compact toric Calabi–Yau manifold Y. □

Remark. We learned from Bong Lian that, in unpublished work, he and Chien-Hao Liu have established mirror theorems for non-compact toric Calabi–Yau manifolds using the arguments of [26]. Once again, the proof for compact toric manifolds applies also to the non-compact toric Calabi–Yau case without significant change. This gives an alternative proof of the second part of Proposition 2.

We can determine the changes of variables in Proposition 2 by expanding the I-functions and the J-functions as Laurent series in z^{-1}. We have

$$J_{\mathcal{X}}(u,z) = z + u_0\delta_0 + u_1\delta_1 + \cdots + u_{n-1}\delta_{n-1} + O(z^{-1})$$

and

$$I_{\mathcal{X}}(x,z) = z + f_0(x)\delta_0 + f_1(x)\delta_1 + \cdots + f_{n-1}(x)\delta_{n-1} + O(z^{-1})$$

where $f_0(x) = x_0$ and for $1 \le k < n$,

$$f_k(x) = \sum_{\substack{\widehat{\beta} \text{ effective:} \\ i(\widehat{\beta}) = k}} \frac{\Gamma\left(1 - \frac{k}{n}\right)}{\Gamma\left(1 + \widehat{\beta}(0)\right)} \frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(1 + \widehat{\beta}(n)\right)} \frac{x_1^{\widehat{\beta}(1)} \cdots x_{n-1}^{\widehat{\beta}(n-1)}}{\widehat{\beta}(1)! \cdots \widehat{\beta}(n-1)!}.$$

The change of variables which equates $I_{\mathcal{X}}$ and $J_{\mathcal{X}}$ is therefore $u_i = f_i(x)$, $0 \leq i < n$. As

$$f_k(x) = x_k + \text{quadratic and higher order terms in } x_1, \ldots, x_{n-1}$$

the functions $f_0(x), \ldots, f_{n-1}(x)$ define co-ordinates on a neighbourhood of the large-radius limit point for $\mathcal{X}$ in $\mathcal{M}_B$. We call these *flat co-ordinates for* $\mathcal{X}$. Similarly,

$$J_Y(t, z) = z + t_0\gamma_0 + t_1\gamma_1 + \cdots + t_{n-1}\gamma_{n-1} + O(z^{-1})$$

and

$$I_Y(y, z) = z + g_0(y)\gamma_0 + g_1(y)\gamma_1 + \cdots + g_{n-1}(y)\gamma_{n-1} + O(z^{-1})$$

for some functions $g_0(y), \ldots, g_{n-1}(y)$ with $g_0(y) = y_0$ and for $1 \leq k < n$,

$$g_k(y) = \log y_k + \text{single-valued analytic function of } y_1, \ldots, y_{n-1}.$$

The change of variables which equates I_Y and J_Y is $t_i = g_i(y)$, $0 \leq i < n$. The functions $g_0(y), \ldots, g_{n-1}(y)$ define multi-valued co-ordinates on a neighbourhood of the large-radius limit point for Y; these are the *flat co-ordinates for* Y. Note that the exponentiated flat co-ordinates $\exp(g_k(y))$ are single-valued.

The J-functions satisfy differential equations which determine the quantum products.

Proposition 3.

(1)
$$z\frac{\partial}{\partial u_i} z\frac{\partial}{\partial u_j} J_{\mathcal{X}}(u, z) = \sum_{k=0}^{n-1} \left(\delta_i \underset{\text{big}}{\star}\right)_j^k z\frac{\partial}{\partial u_k} J_{\mathcal{X}}(u, z)$$

where $\left(\delta_i \underset{\text{big}}{\star}\right)_j^k$ *are the matrix entries of the product* (5).

(2)
$$z\frac{\partial}{\partial t_i} z\frac{\partial}{\partial t_j} J_Y(t, z) = \sum_{k=0}^{n-1} \left(\gamma_i \underset{\text{big}}{\star}\right)_j^k z\frac{\partial}{\partial t_k} J_Y(t, z)$$

where $\left(\gamma_i \underset{\text{big}}{\star}\right)_j^k$ *are the matrix entries of the product* (6).

Proof. Part (2) is well-known to experts (*cf.* [19, Chapter 10; 28, Proposition 2]). By the Divisor Equation, $z\frac{\partial}{\partial t_i} J_Y(t, z)$ is equal to

$$z\, e^{t_0/z} e^{(t_1\gamma_1 + \cdots + t_{n-1}\gamma_{n-1})/z} \left(\gamma_i + \sum_\beta \sum_{k=0}^{n-1} \left\langle \gamma_i, \frac{\gamma_k}{z - \psi_1} \right\rangle_\beta^Y e^{d_1 t_1 + \cdots + d_{n-1}t_{n-1}} \gamma^k \right)$$

and $z\frac{\partial}{\partial t_i} z\frac{\partial}{\partial t_j} J_Y(t, z)$ is equal to

$$z^2\, e^{t_0/z} e^{(t_1\gamma_1 + \cdots + t_{n-1}\gamma_{n-1})/z} \sum_\beta \sum_{k=0}^{n-1} \left\langle \gamma_i, \gamma_j, \frac{\gamma_k}{z - \psi_1} \right\rangle_\beta^Y e^{d_1 t_1 + \cdots + d_{n-1}t_{n-1}} \gamma^k.$$

This last expression is

$$z\, e^{t_0/z} e^{(t_1\gamma_1+\cdots+t_{n-1}\gamma_{n-1})/z} \sum_\beta \sum_{k=0}^{n-1} \langle \gamma_i, \gamma_j, \gamma_k \rangle_\beta^Y \, e^{d_1 t_1 + \cdots + d_{n-1}t_{n-1}} \gamma^k$$

$$+ z\, e^{t_0/z} e^{(t_1\gamma_1+\cdots+t_{n-1}\gamma_{n-1})/z} \sum_\beta \sum_{k=0}^{n-1} \sum_{m\geq 1} \left\langle \gamma_i, \gamma_j, \frac{\gamma_k \psi_1^m}{z^m} \right\rangle_\beta^Y e^{d_1 t_1 + \cdots + d_{n-1}t_{n-1}} \gamma^k.$$

Applying the Topological Recursion Relations [28, Equation 6] yields

$$z\, e^{t_0/z} e^{(t_1\gamma_1+\cdots+t_{n-1}\gamma_{n-1})/z} \sum_\beta \sum_{k=0}^{n-1} \langle \gamma_i, \gamma_j, \gamma_k \rangle_\beta^Y \, e^{d_1 t_1 + \cdots + d_{n-1}t_{n-1}} \gamma^k$$

$$+ z\, e^{t_0/z} e^{(t_1\gamma_1+\cdots+t_{n-1}\gamma_{n-1})/z} \sum_\beta \sum_{\beta'+\beta''=\beta} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \langle \gamma_i, \gamma_j, \gamma^l \rangle_{\beta'}^Y \times \left\langle \gamma_l, \frac{\gamma_k}{z-\psi_1} \right\rangle_{\beta''}^Y e^{d_1 t_1 + \cdots + d_{n-1}t_{n-1}} \gamma^k$$

and this is

$$\left(\sum_{\beta'} \sum_{l=0}^{n-1} \langle \gamma_i, \gamma_j, \gamma^l \rangle_{\beta'}^Y \, e^{d'_1 t_1 + \cdots + d'_{n-1} t_{n-1}} \right) z \frac{\partial}{\partial t_l} J_Y(t,z),$$

where the sum in parentheses is over $\beta' = d'_1 \beta_1 + \cdots + d'_{n-1}\beta_{n-1}$ with each $d'_i \geq 0$. This proves (2). The proof of (1) is essentially identical, but uses the fake Divisor Equation [8, Section 2.2] instead of the Divisor Equation and the Topological Recursion Relations for orbifolds [32, Section 2.5.5] instead of the Topological Recursion Relations for varieties. □

4.2. **From PF to QC.** Propositions 2 and 3 together show that we can determine the quantum products (5) and (6) by looking at the differential equations satisfied by $I_{\mathcal{X}}$ and I_Y *in flat co-ordinates*:

$$z \frac{\partial}{\partial u_i} z \frac{\partial}{\partial u_j} I_{\mathcal{X}}(x(u), z) = \sum_{k=0}^{n-1} \left(\delta_i \underset{\text{big}}{\star} \right)_j^k z \frac{\partial}{\partial u_k} I_{\mathcal{X}}(x(u), z) \tag{10}$$

$$z \frac{\partial}{\partial t_i} z \frac{\partial}{\partial t_j} I_Y(y(t), z) = \sum_{k=0}^{n-1} \left(\gamma_i \underset{\text{big}}{\star} \right)_j^k z \frac{\partial}{\partial t_k} I_Y(y(t), z) \tag{11}$$

A more invariant way to say this is as follows. Let λ_1, λ_2 be fixed complex numbers. If we associate to a vector field $v = \sum v_k(y) \frac{\partial}{\partial y_k}$ on $\mathcal{M}_B$ the differential operator $\sum z v_k(y) \frac{\partial}{\partial y_k}$ then the systems of differential equations (8), (9) define a $\mathcal{D}$-module on $\mathcal{M}_B$. The characteristic variety $\mathfrak{V}$ of this $\mathcal{D}$-module is a subscheme of $T^\star \mathcal{M}_B$, and we can read off the quantum products from the algebra of functions $\mathcal{O}_{\mathfrak{V}}$. Indeed, choosing flat co-ordinates on a neighbourhood U of the large-radius limit point for $\mathcal{X}$ in $\mathcal{M}_B$ identifies $\mathcal{O}_U$ with analytic functions in $u_0, \ldots, u_{n-1}$ and identifies the algebra of fiberwise-polynomial functions on $T^\star U$ with $\mathcal{O}_U[\xi_0, \ldots, \xi_{n-1}]$; here ξ_k is the fiberwise-linear function on $T^\star U$ given by $\frac{\partial}{\partial u_k}$. The ideal defining the characteristic variety $\mathfrak{V}$ is generated by elements

$$P(u_0, \ldots, u_{n-1}, \xi_0, \ldots, \xi_{n-1}, 0)$$

where $P(u_0, \ldots, u_{n-1}, \xi_0, \ldots, \xi_{n-1}, z)$ runs over the set of fiberwise-polynomial functions on $T^\star U$ which depend polynomially on z and satisfy

$$P\left(u_0, \ldots, u_{n-1}, z\frac{\partial}{\partial u_0}, \ldots, z\frac{\partial}{\partial u_{n-1}}, z\right) I_{\mathcal{X}}(u, z) = 0.$$

Equation (10) implies that

$$\mathcal{O}_{\mathfrak{V}}|_U = \mathcal{O}_U[\xi_0, \ldots, \xi_{n-1}]/\mathfrak{I}$$

where the ideal $\mathfrak{I}$ is generated by

$$\xi_i \xi_j = \sum_{k=0}^{n-1} \left(\delta_i \underset{\text{big}}{\star}\right)_j^k \xi_k \qquad\qquad 0 \leq i, j < n.$$

In other words, the quantum cohomology algebra (5) of $\mathcal{X}$ is the algebra of functions $\mathcal{O}_{\mathfrak{V}}|_U$ on the characteristic variety $\mathfrak{V}$, *written in flat co-ordinates on* U.

Similarly, choosing flat co-ordinates on a neighbourhood V of the large-radius limit point for Y in $\mathcal{M}_B$ identifies $\mathcal{O}_V$ with analytic functions in $t_0, \ldots, t_{n-1}$, and identifies the algebra of fiberwise-polynomial functions on $T^\star V$ with $\mathcal{O}_V[\eta_0, \ldots, \eta_{n-1}]$ where η_k is the fiberwise-linear function on $T^\star V$ given by $\frac{\partial}{\partial t_k}$. Equation (11) implies that

$$\mathcal{O}_{\mathfrak{V}}|_V = \mathcal{O}_V[\eta_0, \ldots, \eta_{n-1}]/\mathfrak{J}$$

where the ideal $\mathfrak{J}$ is generated by

$$\eta_i \eta_j = \sum_{k=0}^{n-1} \left(\gamma_i \underset{\text{big}}{\star}\right)_j^k \eta_k \qquad\qquad 0 \leq i, j < n,$$

and so the quantum cohomology algebra (6) of Y is the algebra of functions $\mathcal{O}_{\mathfrak{V}}|_V$ on the characteristic variety $\mathfrak{V}$, *written in flat co-ordinates on* V.

The characteristic variety $\mathfrak{V}$ is a global analytic object — $\mathcal{O}_{\mathfrak{V}}$ gives an analytic sheaf of $\mathcal{O}_{\mathcal{M}_B}$-algebras, defined over all of $\mathcal{M}_B$ — so to show that the quantum cohomology algebras of $\mathcal{X}$ and of Y are related by analytic continuation followed by the change-of-variables

$$t_i = \begin{cases} u_0, & i = 0 \\ -\frac{2\pi\sqrt{-1}}{n} + \sum_{j=1}^{n-1} L_{ij} u_j, & i > 0 \end{cases}$$

we just need to show that the flat co-ordinates for $\mathcal{X}$ and for Y are related by analytic continuation followed by the change-of-variables

$$g_i(y) = \begin{cases} f_0(x), & i = 0 \\ -\frac{2\pi\sqrt{-1}}{n} + \sum_{j=1}^{n-1} L_{ij} f_j(x), & i > 0. \end{cases}$$

Proposition 4. *There exists a path from the large-radius limit point for Y to the large-radius limit point for $\mathcal{X}$ such that the analytic continuation of the flat co-ordinates $g_i(y)$, $1 \leq i < n$, along that path satisfy*

$$g_i(y) = -\frac{2\pi\sqrt{-1}}{n} + \frac{1}{n}\sum_{k=1}^{n-1} \zeta^{2ki} \left(\zeta^{-k} - \zeta^k\right) f_k(x),$$

where $\zeta = \exp\left(\frac{\pi\sqrt{-1}}{n}\right)$.

Proof. The flat co-ordinates $f_1(x), \ldots, f_{n-1}(x)$ and $g_1(y), \ldots, g_{n-1}(y)$ are independent of λ_1, λ_2, x_0, and y_0, so they can be extracted from the z^0 terms of $I_{\mathcal{X}}$ and I_Y after setting $\lambda_1 = \lambda_2 = x_0 = y_0 = 0$. But $I_{\mathcal{X}}|_{\lambda_1=\lambda_2=x_0=0}$ and $I_Y|_{\lambda_1=\lambda_2=y_0=0}$ satisfy the systems of differential equations (8a), (9a) with λ_1 and λ_2 set to zero, and once λ_1 and λ_2 are set to zero the z-dependence in these differential equations cancels. The flat co-ordinates $f_1(x), \ldots, f_{n-1}(x)$ and $g_1(y), \ldots, g_{n-1}(y)$ therefore satisfy

$$\left(\prod_{j:D_j(\beta)>0} \prod_{m=0}^{D_j(\beta)-1} (\daleth_j - m)\right) f = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}} \left(\prod_{j:D_j(\beta)<0} \prod_{m=0}^{-D_j(\beta)-1} (\daleth_j - m)\right) f \tag{12}$$

for every $\beta = d_1\beta_1 + \cdots + d_{n-1}\beta_{n-1}$, where

$$\daleth_j = \begin{cases} y_1 \frac{\partial}{\partial y_1} & j = 0 \\ -2y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} & j = 1 \\ y_{j-1} \frac{\partial}{\partial y_{j-1}} - 2y_j \frac{\partial}{\partial y_j} + y_{j+1} \frac{\partial}{\partial y_{j+1}} & 1 < j < n-1 \\ y_{n-2} \frac{\partial}{\partial y_{n-2}} - 2y_{n-1} \frac{\partial}{\partial y_{n-1}} & j = n-1 \\ y_{n-1} \frac{\partial}{\partial y_{n-1}} & j = n. \end{cases}$$

This is the GKZ system associated to Y. It has rank n, and both $f_1(x), \ldots, f_{n-1}(x)$ plus the constant function and $g_1(y), \ldots, g_{n-1}(y)$ plus the constant function form bases of solutions. Any analytic continuation $\tilde{g}_i(y)$ of $g_i(y)$ to a neighbourhood of the large-radius limit point for $\mathcal{X}$ still satisfies (12), so

$$\tilde{g}_i(y) = \sum_{j=1}^{n-1} L_{ij} f_j(x) + m_i$$

for some constants L_{ij} and m_i. Thus any analytic continuation of $g_i(y)$ is an affine-linear combination of the flat co-ordinates $f_1(x), \ldots, f_{n-1}(x)$. It remains to choose a specific analytic continuation and determine the corresponding constants L_{ij} and m_i.

We proved in [17] that another basis of solutions to the GKZ system (12) is given by the constant function together with

$$\log \kappa_i(x) - \log \kappa_{i-1}(x), \qquad 1 \le i < n,$$

where $\kappa_i(x)$ are roots of the polynomial

$$W_{\mathcal{X}}(\kappa) = \kappa^n + x_{n-1}\kappa^{n-1} + x_{n-2}\kappa^{n-2} + \cdots + x_1\kappa + 1.$$

We number the roots such that as $x \to 0$,

$$\kappa_i(x) \to \zeta^{2i+1}, \qquad 0 \le i < n.$$

Each $\log \kappa_i(x)$, $0 \le i < n$, is also a solution to (12). Equation 25 in [17] gives

$$\log \kappa_i(x) = \frac{(2i+1)\pi\sqrt{-1}}{n} + \frac{1}{n} \sum_{k=1}^{n-1} \zeta^{(2i+1)k} f_k(x). \tag{13}$$

Consider also the polynomial

$$W_Y(\mu) = \mu^n + \mu^{n-1} + y_1\mu^{n-2} + y_1^2 y_2 \mu^{n-3} + y_1^3 y_2^2 y_3 \mu^{n-4} + \cdots \\ + y_1^{n-1} y_2^{n-2} \cdots y_{n-2}^2 y_{n-1}$$

and number its roots $\mu_i(y)$, $0 \le i < n$ such that as $y \to 0$

$$\begin{aligned} \mu_0(y) &\to -1 \\ \mu_1(y) &\sim -y_1 \\ \mu_2(y) &\sim -y_1 y_2 \\ &\vdots \\ \mu_{n-1}(y) &\sim -y_1 y_2 \cdots y_{n-1}. \end{aligned}$$

We have $W_{\mathcal{X}}(\kappa) = 0$ if and only if $W_Y(1/(x_1\kappa)) = 0$, where x_i and y_j are related by (1), so still another basis of solutions to the GKZ system (12) is

$$\log \mu_i(y) - \log \mu_{i-1}(y) \qquad\qquad 1 \le i < n$$

together with the constant function. The solution $g_i(y)$ is singled out by its behaviour $g_i(y) = \log y_i + O(y_1, \ldots, y_{n-1})$ as $y \to 0$, so

$$g_i(y) = \log \mu_i(y) - \log \mu_{i-1}(y).$$

Along any path from the large-radius limit point for Y to the large-radius limit point for $\mathcal{X}$, the root $\mu_i(y)$ of W_Y analytically continues to the root $1/(x_1\kappa_{\sigma(i)}(x))$ of $W_{\mathcal{X}}$, for some permutation σ of $\{0, 1, \ldots, n-1\}$. The group of monodromies around the discriminant locus of $W_{\mathcal{X}}$ acts n-transitively on the set of roots of $W_{\mathcal{X}}$, so we can choose a path such that σ is the identity permutation. Along this path, $\log \mu_i(y) - \log \mu_{i-1}(y)$ analytically continues to $\log \kappa_{i-1}(x) - \log \kappa_i(x)$, $1 \le i < n$. Applying equation (13) yields

$$g_i(y) = -\frac{2\pi\sqrt{-1}}{n} + \frac{1}{n}\sum_{k=1}^{n-1} \zeta^{2ki}\left(\zeta^{-k} - \zeta^k\right) f_k(x).$$

□

Remark. For an explicit path satisfying the conditions in Proposition 4, we can concatenate two paths defined as follows. The first runs from $(y_0, y_1, \ldots, y_{n-1}) = (0, 0, \ldots, 0, 0)$ to $(y_0, y_1, \ldots, y_{n-1}) = (0, 1, 1, \ldots, 1, 1)$ and is given by $y_0 = 0$ and

$$W_Y(\mu) = \Big(\mu - \left(-1 - \epsilon\rho^2 - \epsilon^2\rho^3 - \ldots - \epsilon^{n-1}\rho^n\right)\Big) \prod_{k=1}^{n-1} \left(\mu - \epsilon^k \rho^{k+1}\right), \quad 0 \le \epsilon \le 1,$$

where $\rho = \exp\left(\frac{2\pi\sqrt{-1}}{n+1}\right)$. The second runs from $(x_0, x_1, \ldots, x_{n-1}) = (0, 1, 1, \ldots, 1, 1)$ to $(x_0, x_1, \ldots, x_{n-1}) = (0, 0, \ldots, 0, 0)$, and is given by $x_0 = 0$ and

$$W_{\mathcal{X}}(\kappa) = \prod_{k=0}^{n-1} \left(\kappa - \exp\left(\pi\sqrt{-1}\left[\frac{2k+1}{n}\epsilon' + \frac{2(n-k)}{n+1}(1-\epsilon')\right]\right)\right), \quad 0 \le \epsilon' \le 1.$$

Note that the points $(y_0, y_1, \ldots, y_{n-1}) = (0, 1, 1, \ldots, 1, 1)$ and $(x_0, x_1, \ldots, x_{n-1}) = (0, 1, 1, \ldots, 1, 1)$ coincide.

4.3. **The Proof of Theorem 1.** Combining Proposition 4 with the discussion at the end of Section 4.2 shows that the quantum cohomology algebras of $\mathcal{X}$ and Y coincide after analytic continuation along the path specified in Proposition 4 followed by the affine-linear change-of-variables

$$t_i = \begin{cases} u_0, & i = 0 \\ -\frac{2\pi\sqrt{-1}}{n} + \sum_{j=1}^{n-1} L_{ij} u_j, & i > 0, \end{cases} \qquad L_{ij} = \frac{\zeta^{2ij}\left(\zeta^{-j} - \zeta^{j}\right)}{n},$$

and the linear isomorphism

$$\begin{aligned} L : H(\mathcal{X}) &\to H(Y) \\ \delta_0 &\mapsto \gamma_0, \\ \delta_j &\mapsto \sum_{i=1}^{n-1} L_{ij}\gamma_i, \qquad 1 \leq j < n. \end{aligned}$$

To see that L preserves the Poincaré pairings, first observe that the bases

$$n\lambda_1\lambda_2, \gamma_1, \gamma_2, \ldots, \gamma_{n-1} \qquad \text{and} \qquad 1, \omega_1, \omega_2, \ldots, \omega_{n-1}$$

for $H(Y)$ are dual with respect to the Poincaré pairing on $H(Y)$. Let $L^\dagger$ denote the adjoint to L with respect to the Poincaré pairing $(\cdot,\cdot)_Y$ and the orbifold Poincaré pairing $(\cdot,\cdot)_{\mathcal{X}}$. It suffices to show that $(L^\dagger\gamma, L^\dagger\gamma')_{\mathcal{X}} = (\gamma,\gamma')_Y$ for all $\gamma, \gamma' \in H(Y)$. For $1 \leq i < n$, we have $(L^\dagger\omega_i, \delta_k)_{\mathcal{X}} = (\omega_i, L\delta_k)_Y = L_{ik}$, and so

$$L^\dagger\omega_i = n\sum_{k=1}^{n-1} L_{ik}\delta_{n-k}, \qquad 1 \leq i < n.$$

Also $L^\dagger 1 = \delta_0$. Straightforward calculation now gives $(L^\dagger 1, L^\dagger 1)_{\mathcal{X}} = (n\lambda_1\lambda_2)^{-1}$, $(L^\dagger 1, L^\dagger\omega_i)_{\mathcal{X}} = 0$ for $1 \leq i < n$, and

$$(L^\dagger\omega_i, L^\dagger\omega_j)_{\mathcal{X}} = \begin{cases} 0 & \text{if } |i-j| > 1 \\ 1 & \text{if } |i-j| = 1 \\ -2 & \text{if } i = j \end{cases} \qquad \text{for } 1 \leq i,j < n.$$

As the class ω_j is the T-equivariant Poincaré-dual to the jth exceptional divisor we see that $L^\dagger$, and hence L, is pairing-preserving. This completes the proof of Theorem 1. □

Remark. A more conceptual explanation of this result is as follows. One can construct a Frobenius manifold from a *variation of semi-infinite Hodge structure* [3] (henceforth $\mathrm{V}\frac{\infty}{2}\mathrm{HS}$) together with a choice of *opposite subspace*[3]. We have argued elsewhere that in certain toric examples one can construct the Frobenius manifold which is the "mirror partner" to the quantum cohomology of Y from a $\mathrm{V}\frac{\infty}{2}\mathrm{HS}$ parameterized by the B-model moduli space of Y, together with a distinguished opposite subspace associated to the large-radius limit point for Y [16]. (The Frobenius manifold mirror to the quantum cohomology of a toric orbifold $\mathcal{X}$ birational to Y is given by the *same* $\mathrm{V}\frac{\infty}{2}\mathrm{HS}$ but the opposite subspace corresponding to the large-radius limit point for $\mathcal{X}$.) One can apply this construction here to get a $\mathrm{V}\frac{\infty}{2}\mathrm{HS}$ parametrized by $\mathcal{M}_B$. This $\mathrm{V}\frac{\infty}{2}\mathrm{HS}$ has the special property that

[3]Mirror symmetry often associates to the quantum cohomology of some target space a "mirror family" of manifolds. In this case one can think of the $\mathrm{V}\frac{\infty}{2}\mathrm{HS}$ as an analog of the usual variation of Hodge structure on the mirror family, and the opposite subspace as an analog of the weight filtration.

the opposite subspace at the large-radius limit point for Y agrees with the opposite subspace at the large-radius limit point for[4] $\mathcal{X}$. In general the difference between the opposite subspaces at different large radius limit points will be measured by an element of Givental's linear symplectic group, but in this case the corresponding group element maps the opposite subspaces isomorphically to each other. This means that we get a Frobenius manifold *over the whole (non-linear) space* $\mathcal{M}_B$. One can construct flat co-ordinates in a neighbourhood of any point of $\mathcal{M}_B$, and the transition functions between such flat co-ordinate patches, such as

$$g_i(y) = \sum_j L_{ij} f_j(x) + \log c_i,$$

are necessarily affine-linear and (Poincaré) metric-preserving.

Remark. It is clear from the proof of Proposition 4 that changing the path along which analytic continuation is taken will result in a corresponding change in the statements of Theorem 1 and its Corollary. Hence the co-ordinate change in Theorem 1 is not unique. This ambiguity can be understood as an automorphism of quantum cohomology. The orbifold fundamental group

$$G := \pi_1^{\mathrm{orb}} \left(\mathcal{M}_B \setminus \{\text{discriminant locus of } W_{\mathcal{X}}\} \right)$$

acts simply-transitively on the set of homotopy types of paths from the large-radius limit point for Y to that for $\mathcal{X}$, and in particular acts transitively on the set of all possible co-ordinate changes obtained by analytic continuation (although this action is not effective). This deserves further study: we just note here the intriguing fact that G is isomorphic to $\widetilde{A}_{n-1} \rtimes \mu_n$, which also appears as a subgroup (generated by spherical twists and line bundles) of the group of autoequivalences of $D^b_Z(Y)$ [5, 24, 25]. Here $\widetilde{A}_{n-1}$ is the affine braid group and $D^b_Z(Y)$ is the bounded derived category of coherent sheaves on Y supported on the exceptional set Z.

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